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A NOTE ON DEGENERATE STIRLING NUMBERS OF THE FIRST KIND ASSOCIATED WITH DEGENERATE GENERALIZED HARMONIC NUMBERS

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ABSTRACT. In this paper, we introduce a new generalization of the degenerate harmonic numbers, which we call the generalized degenerate harmonic numbers. For these numbers, we find explicit expressions, and some related finite sums that evaluate to simple integers. Furthermore, we express the degenerate harmonic numbers in terms of the newly defined generalized numbers, the unsigned degenerate Stirling numbers of the first kind and the degenerate Bernoulli numbers, and we provide a closed-form expression for a specific sum involving degenerate hyperharmonic numbers.

1. INTRODUCTION

Recent investigations have explored degenerate versions of many special numbers and polynomials (see [8,12-15,17,18,20]), a field initiated by Carlitz's work on degenerate Bernoulli and Euler polynomials (see [4]). This line of inquiry has since been expanded to include transcendental functions and umbral calculus, leading to the development of the degenerate gamma function (see [16]) and the degenerate umbral calculus (see [10]).

This paper presents several new results regarding the degenerate harmonic numbers, the generalized degenerate harmonic numbers, the degenerate hyperharmonic numbers and some related identities. We begin by reviewing the definitions of degenerate exponentials and degenerate logarithms. We then recall the unsigned degenerate Stirling numbers of the first kind, denoted as ${n \brack k}_\lambda$, and the degenerate Stirling numbers of the second kind, ${n \brace k}_\lambda$. Following this, we refresh the reader's memory on harmonic numbers, degenerate harmonic numbers, hyperharmonic numbers, and degenerate hyperharmonic numbers. Section 2 contains the core findings of this research. We introduce a new generalization of the degenerate harmonic numbers, $H_{n,\lambda}$, which we call the generalized degenerate harmonic numbers, $H_\lambda(n,r)$, for $r \geq 0$ and $n \geq r+1$. Note that when $r=0$, this new number simplifies to the original degenerate harmonic number, i.e., $H_\lambda(n,0) = H_{n,\lambda}$.
We provide several key theorems:
• Theorem 2.1 and Theorem 2.2: These theorems express $H_\lambda(n,r)$ as a finite sum. Theorem 2.1's expression involves products of binomial coefficients, while Theorem 2.2's expression involves the unsigned degenerate Stirling numbers of the first kind, ${m \brack r+1}_\lambda$.
• Theorem 2.3 and Theorem 2.4: These theorems present two finite sums. Theorem 2.3 shows that a sum involving $H_\lambda(n,r)$ and another involving ${m \brack r+1}_\lambda$ both

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sum to n. Similarly, Theorem 2.4 demonstrates that two related sums, one involving $H_\lambda(n,r)$ and the other $\left[\begin{smallmatrix} m \\ r+1 \end{smallmatrix}\right]_\lambda$ both sum to 1.
• Theorem 2.5: This theorem expresses $H_{n,\lambda}$ as two finite sums. One sum involves $H_\lambda(n+1,r)$, and the other involves $\left[\begin{smallmatrix} m \\ r+1 \end{smallmatrix}\right]_\lambda$.
• Theorem 2.6: Here, we express $H_{n,\lambda}$ as two different finite sums: one involving the degenerate Bernoulli numbers and $H_\lambda(n-1,r)$, and the other involving the degenerate Bernoulli numbers and $\left[\begin{smallmatrix} m \\ r+1 \end{smallmatrix}\right]_\lambda$.
• Theorem 2.7: The final theorem provides a closed-form expression for the sum $\sum_{r=1}^{n}(-1)^r rH_{n-r+1,\lambda}^{(r)}$, where $H_{n-r+1,\lambda}^{(r)}$ are the degenerate hyperharmonic numbers.
Finally, Section 3 provides a summary of our results and concludes the paper.

A list of general references can be found in [1,6,21]. In the rest of this section, we recall the facts that are needed throughout this paper.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are given by

$$e_\lambda^x(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad e_\lambda(t) = e_\lambda^1(t), \quad (\text{see } [8,10,16]),$$

where

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1).$$

Note that

$$\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}.$$

The degenerate logarithm $\log_\lambda(t)$ is defined as the compositional inverse of $e_\lambda(t)$, and hence we have (see [8,10,16])

$$(1) \qquad \log_\lambda(1+t) = \frac{1}{\lambda}\left((1+t)^\lambda - 1\right) = \sum_{n=1}^{\infty}\binom{\lambda-1}{n-1}\frac{t^n}{n}.$$

The degenerate Stirling numbers of the first kind $S_{1,\lambda}(n,k)$ are given by

$$(2) \qquad (x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k)\,(x)_{k,\lambda}, \quad (n \geq 0), \quad (\text{see } [14,15,17]),$$

where

$$(x)_0 = 1, \quad (x)_n = x(x-1)(x-2)\cdots(x-n+1), \quad (n \geq 1).$$

The unsigned degenerate Stirling numbers of the first kind are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_\lambda = (-1)^{n-k}S_{1,\lambda}(n,k),$$

where n,k are nonnegative integers with $n \geq k$. From (1), we note that

$$(3) \qquad \frac{1}{k!}\log_{-\lambda}^k\left(\frac{1}{1-t}\right) = \sum_{n=k}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix}_\lambda \frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see } [14]).$$

As the inversion formula of (2), the degenerate Stirling numbers of the second kind $\left\{\begin{matrix} n \\ k \end{matrix}\right\}_\lambda$ are defined by

$$(4) \qquad (x)_{n,\lambda} = \sum_{k=0}^{n}\left\{\begin{matrix} n \\ k \end{matrix}\right\}_\lambda (x)_k, \quad (n \geq 0), \quad (\text{see } [14,15,17]).$$

From (4), we note that

$$\frac{1}{k!}\left(e_{\lambda}(t)-1\right)^{k}=\sum_{n=k}^{\infty}\begin{Bmatrix} n \\ k \end{Bmatrix}_{\lambda}\frac{t^{n}}{n!},\quad (n\geq 0),\quad (\text{see [14]}).$$

It is well known that the harmonic numbers are defined by (see [3,5,9])

$$H_0=0,\quad H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}. \tag{5}$$

Thus, by (5), we get (see [11,19])

$$\frac{1}{1-t}\log\left(\frac{1}{1-t}\right)=\sum_{n=1}^{\infty}H_n t^n.$$

Recently, Kim–Kim introduced the degenerate harmonic numbers given by

$$H_{0,\lambda}=0,\quad H_{n,\lambda}=\frac{1}{\lambda}\sum_{k=1}^{n}\binom{\lambda}{k}(-1)^{k-1}=\sum_{k=1}^{n}\binom{\lambda-1}{k-1}\frac{(-1)^{k-1}}{k}, \tag{6}$$

where n is a positive integer (see [8,13,14]). Note that

$$\lim_{\lambda\to 0}H_{n,\lambda}=H_n,\quad (n\geq 0).$$

From (6), we note that

$$\frac{1}{1-t}\log_{-\lambda}\left(\frac{1}{1-t}\right)=\sum_{n=1}^{\infty}H_{n,\lambda}t^n,\quad H_{0,\lambda}=0,\quad (\text{see [17,18,20]}). \tag{7}$$

In 1996, Conway and Guy introduced the hyperharmonic numbers $H_n^{(r)}$, $(n,r\geq 0)$, which are defined by (see [2,7,11])

$$H_0^{(r)}=0,\ (r\geq 0),\quad H_n^{(0)}=\frac{1}{n},\ (n\geq 1),\quad H_n^{(r)}=\sum_{k=1}^{n}H_k^{(r-1)},\quad (n,r\geq 1). \tag{8}$$

Then, by (8), we get

$$\frac{1}{(1-t)^r}\log\left(\frac{1}{1-t}\right)=\sum_{n=1}^{\infty}H_n^{(r)}t^n,\quad (\text{see [2,11]}).$$

Recently, Kim–Kim introduced the degenerate hyperharmonic numbers given by (see [8,14,17])

$$\begin{aligned}
&H_{0,\lambda}^{(r)}=0,\\
&H_{n,\lambda}^{(0)}=\frac{\lambda^{n-1}}{n!}(-1)^{n-1}(1)_{n,1/\lambda}=\frac{1}{\lambda}\binom{\lambda}{n}(-1)^{n-1},\quad (n\geq 1),\\
&H_{n,\lambda}^{(r)}=\sum_{k=1}^{n}H_{n,\lambda}^{(r-1)},\quad (n,r\geq 1).
\end{aligned} \tag{9}$$

From (9), we obtain

$$\frac{1}{(1-t)^r}\log_{-\lambda}\left(\frac{1}{1-t}\right)=\sum_{n=1}^{\infty}H_{n,\lambda}^{(r)}t^n. \tag{10}$$

2. A generalization of degenerate harmonic numbers

For any integer $r \geq 0$, we define the *generalized degenerate harmonic numbers* by

$$\frac{1}{1-x}\log_{-\lambda}^{r+1}\left(\frac{1}{1-x}\right)=\sum_{n=r+1}^{\infty}H_{\lambda}(n,r)x^{n}. \tag{11}$$

Note that

$$H_{n,\lambda}(n,0)=H_{n,\lambda}, \quad (n \geq 0).$$

From (11), we note that

$$\begin{aligned}
&\frac{1}{1-x}\log_{-\lambda}^{r+1}\left(\frac{1}{1-x}\right)\\
&=\frac{1}{1-x}\underbrace{\log_{-\lambda}\left(\frac{1}{1-x}\right)\times\log_{-\lambda}\left(\frac{1}{1-x}\right)\times\cdots\times\log_{-\lambda}\left(\frac{1}{1-x}\right)}_{\text{(r+1)-times}}\\
&=\frac{1}{1-x}\sum_{l_0=1}^{\infty}\binom{\lambda-1}{l_0-1}\frac{(-1)^{l_0-1}}{l_0}x^{l_0}\sum_{l_1=1}^{\infty}\binom{\lambda-1}{l_1-1}\frac{(-1)^{l_1-1}}{l_1}x^{l_1}\cdots\sum_{l_r=1}^{\infty}\binom{\lambda-1}{l_r-1}\frac{(-1)^{l_r-1}}{l_r}x^{l_r}\\
&=\sum_{k=0}^{\infty}x^{k}\sum_{m=r+1}^{\infty}\sum_{l_0+l_1+\cdots+l_r=m}\frac{\binom{\lambda-1}{l_0-1}\binom{\lambda-1}{l_1-1}\cdots\binom{\lambda-1}{l_r-1}}{l_0l_1\cdots l_r}(-1)^{m-r-1}x^{m}\\
&=\sum_{n=r+1}^{\infty}\sum_{m=r+1}^{n}\sum_{l_0+l_1+\cdots+l_r=m}\frac{\binom{\lambda-1}{l_0-1}\binom{\lambda-1}{l_1-1}\cdots\binom{\lambda-1}{l_r-1}}{l_0l_1\cdots l_r}(-1)^{m-r-1}x^{n},
\end{aligned} \tag{12}$$

where $l_0,l_1,\ldots,l_r$ run over all positive integers satisfying $l_0+l_1+\cdots+l_r=m$. Therefore, by (11) and (12), we obtain the following theorem.

Theorem 2.1. *For any integers n,r, with $r \geq 0$, $n \geq r+1$, we have*

$$\begin{aligned}
H_{\lambda}(n,r)&=\sum_{m=r+1}^{n}\sum_{l_0+l_1+\cdots+l_r=m}\frac{\binom{\lambda-1}{l_0-1}\binom{\lambda-1}{l_1-1}\cdots\binom{\lambda-1}{l_r-1}}{l_0l_1\cdots l_r}(-1)^{m-r-1}\\
&=\sum_{r+1\leq l_0+l_1+\cdots+l_r\leq n}\frac{(-1)^{l_0-1}\binom{\lambda-1}{l_0-1}(-1)^{l_1-1}\binom{\lambda-1}{l_1-1}\cdots(-1)^{l_r-1}\binom{\lambda-1}{l_r-1}}{l_0l_1\cdots l_r},
\end{aligned}$$

where $l_0,l_1,\ldots,l_r$ run over all positive integers satisfying $r+1 \leq l_0+l_1+\cdots+l_r \leq n$.

From (3), we note that

$$\begin{aligned}
\frac{1}{1-x}\log_{-\lambda}^{r+1}\left(\frac{1}{1-x}\right)&=\frac{(r+1)!}{1-x}\frac{1}{(r+1)!}\log_{-\lambda}^{r+1}\left(\frac{1}{1-x}\right)\\
&=(r+1)!\sum_{l=0}^{\infty}x^{l}\sum_{m=r+1}^{\infty}\begin{bmatrix}m\\r+1\end{bmatrix}_{\lambda}\frac{x^{m}}{m!}\\
&=\sum_{n=r+1}^{\infty}\sum_{m=r+1}^{n}\begin{bmatrix}m\\r+1\end{bmatrix}_{\lambda}\frac{(r+1)!}{m!}x^{n}.
\end{aligned} \tag{13}$$

Therefore, by (11), (13) and Theorem 2.1, we obtain the following theorem.

Theorem 2.2. *For any integers n, r, with $r \geq 0$, $n \geq r+1$, we have*

$$H_\lambda(n,r) = (r+1)! \sum_{m=r+1}^{n} \begin{bmatrix} m \\ r+1 \end{bmatrix}_\lambda \frac{1}{m!}. \tag{14}$$

By (11), we get

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{r=0}^{n-1} \frac{(1)_{r+1,-\lambda}}{(r+1)!} H_\lambda(n,r)x^n &= \sum_{r=0}^{\infty} \frac{(1)_{r+1,-\lambda}}{(r+1)!} \sum_{n=r+1}^{\infty} H_\lambda(n,r)x^n \\
&= \frac{1}{1-x} \sum_{r=0}^{\infty} \frac{(1)_{r+1,-\lambda}}{(r+1)!} \log_{-\lambda}^{r+1}\left(\frac{1}{1-x}\right) \\
&= \frac{1}{1-x}\left(\sum_{r=0}^{\infty} \frac{(1)_{r,-\lambda}}{r!} \log_{-\lambda}^{r}\left(\frac{1}{1-x}\right) - 1\right) \\
&= \frac{1}{1-x}\left(e_{-\lambda}\log_{-\lambda}\left(\frac{1}{1-x}\right) - 1\right) \\
&= \frac{1}{(1-x)^2} - \frac{1}{1-x} \\
&= \sum_{n=0}^{\infty}(n+1)x^n - \sum_{n=0}^{\infty} x^n \\
&= \sum_{n=1}^{\infty} nx^n.
\end{aligned} \tag{15}$$

By comparing the coefficients on both sides of (15), we get

$$\sum_{r=0}^{n-1} \frac{H_\lambda(n,r)}{(r+1)!}(1)_{r+1,-\lambda} = n, \quad (n \geq 1,\ r \geq 0). \tag{16}$$

From (14) and (16), we note that

$$\begin{aligned}
n &= \sum_{r=0}^{n-1}(1)_{r+1,-\lambda}\frac{H_\lambda(n,r)}{(r+1)!} = \sum_{r=0}^{n-1}(1)_{r+1,-\lambda}\sum_{m=r+1}^{n}\frac{\begin{bmatrix} m \\ r+1 \end{bmatrix}_\lambda}{m!} \\
&= \sum_{m=1}^{n}\sum_{r=0}^{m-1}\frac{1}{m!}\begin{bmatrix} m \\ r+1 \end{bmatrix}_\lambda (1)_{r+1,-\lambda}.
\end{aligned} \tag{17}$$

Therefore, by (16) and (17), we obtain the following theorem.

Theorem 2.3. *For $n \geq 1$ and $r \geq 0$, we have*

$$n = \sum_{r=0}^{n-1}\frac{H_\lambda(n,r)}{(r+1)!}(1)_{r+1,-\lambda} = \sum_{m=1}^{n}\sum_{r=0}^{m-1}\frac{\begin{bmatrix} m \\ r+1 \end{bmatrix}_\lambda}{m!}(1)_{r+1,-\lambda}.$$

From (11), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{r=0}^{n-1}\frac{(-1)^r(1)_{r+1,\lambda}}{(r+1)!}H_\lambda(n,r)x^n \\
&=\sum_{r=0}^{\infty}\frac{(-1)^r(1)_{r+1,\lambda}}{(r+1)!}\sum_{n=r+1}^{\infty}H_\lambda(n,r)x^n \\
&=\frac{1}{1-x}\sum_{r=0}^{\infty}\frac{(-1)^r(1)_{r+1,\lambda}}{(r+1)!}\log_{-\lambda}^{r+1}\left(\frac{1}{1-x}\right) \\
&=-\frac{1}{1-x}\sum_{r=0}^{\infty}\frac{(-1)^{r+1}(1)_{r+1,\lambda}}{(r+1)!}\log_{-\lambda}^{r+1}\left(\frac{1}{1-x}\right) \\
&=-\frac{1}{1-x}\left(\sum_{r=0}^{\infty}\frac{(-1)^r(1)_{r,\lambda}}{r!}\log_{-\lambda}^{r}\left(\frac{1}{1-x}\right)-1\right) \\
&=-\frac{1}{1-x}\left(e_\lambda\left(-\log_{-\lambda}\left(\frac{1}{1-x}\right)\right)-1\right) \\
&=-\frac{1}{1-x}\left(e_\lambda\left(\log_\lambda(1-x)\right)-1\right) \\
&=-1+\frac{1}{1-x}=\sum_{n=1}^{\infty}x^n.
\end{aligned}
\tag{18}
$$

By (18), we get

$$
\sum_{r=0}^{n-1}\frac{(-1)^r(1)_{r+1,\lambda}}{(r+1)!}H_\lambda(n,r)=1,\quad (n\geq 1,\ r\geq 0). \tag{19}
$$

From (14) and (19), we have

$$
\begin{aligned}
1&=\sum_{r=0}^{n-1}(-1)^r(1)_{r+1,\lambda}\frac{H_\lambda(n,r)}{(r+1)!}=\sum_{r=0}^{n-1}(-1)^r(1)_{r+1,\lambda}\sum_{m=r+1}^{n}\begin{bmatrix}m\\r+1\end{bmatrix}_\lambda\frac{1}{m!} \\
&=\sum_{m=1}^{n}\sum_{r=0}^{m-1}\begin{bmatrix}m\\r+1\end{bmatrix}_\lambda\frac{1}{m!}(-1)^r(1)_{r+1,\lambda}.
\end{aligned}
\tag{20}
$$

Therefore, by (19) and (20), we obtain the following theorem.

Theorem 2.4. *For $n\geq 1$ and $r\geq 0$, we have*

$$
1=\sum_{r=0}^{n-1}\frac{(-1)^r(1)_{r+1,\lambda}}{(r+1)!}H_\lambda(n,r)=\sum_{m=1}^{n}\sum_{r=0}^{m-1}\begin{bmatrix}m\\r+1\end{bmatrix}_\lambda\frac{1}{m!}(-1)^r(1)_{r+1,\lambda}.
$$

By (7) and (14), we get

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{r=1}^{n}\frac{(-1)^{r+1}(1)_{r,\lambda}}{r!}H_\lambda(n+1,r)x^n=\sum_{r=1}^{\infty}\frac{(-1)^{r+1}(1)_{r,\lambda}}{r!}\sum_{n=r}^{\infty}H_\lambda(n+1,r)x^n\\
&=\frac{1}{x}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r!}(1)_{r,\lambda}\sum_{n=r+1}^{\infty}H_\lambda(n,r)x^n=\frac{1}{x}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r!}(1)_{r,\lambda}\frac{\log_{-\lambda}^{r+1}\left(\frac{1}{1-x}\right)}{1-x}\\
&=\frac{-\log_{-\lambda}\left(\frac{1}{1-x}\right)}{x(1-x)}\sum_{r=1}^{\infty}\frac{(-1)^r}{r!}(1)_{r,\lambda}\log_{-\lambda}^{r}\left(\frac{1}{1-x}\right)\\
&=\frac{\log_\lambda(1-x)}{x(1-x)}\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}(1)_{r,\lambda}\log_{-\lambda}^{r}\left(\frac{1}{1-x}\right)-1\right)\\
&=\frac{\log_\lambda(1-x)}{x(1-x)}\left(e_\lambda\left(-\log_{-\lambda}\left(\frac{1}{1-x}\right)\right)-1\right)\\
&=\frac{\log_\lambda(1-x)}{x(1-x)}\left(e_\lambda\left(\log_\lambda(1-x)\right)-1\right)\\
&=\frac{-\log_\lambda(1-x)}{(1-x)}=\frac{1}{1-x}\log_{-\lambda}\left(\frac{1}{1-x}\right)=\sum_{n=1}^{\infty}H_{n,\lambda}x^n.
\end{aligned}
\tag{21}
$$

By comparing the coefficients on both sides of (21), we obtain the following equation:

$$
\sum_{r=1}^{n}\frac{(-1)^{r+1}}{r!}(1)_{r,\lambda}H_\lambda(n+1,r)=H_{n,\lambda},\quad (n\geq 1).
\tag{22}
$$

On the other hand, by (14), we get

$$
\begin{aligned}
&\sum_{r=1}^{n}\frac{(-1)^{r+1}}{r!}(1)_{r,\lambda}H_\lambda(n+1,r)=\sum_{r=1}^{n}(r+1)(-1)^{r+1}(1)_{r,\lambda}\frac{H_\lambda(n+1,r)}{(r+1)!}\\
&=\sum_{r=1}^{n}(r+1)(-1)^{r+1}(1)_{r,\lambda}\sum_{m=r+1}^{n+1}\begin{bmatrix}m\\r+1\end{bmatrix}_\lambda\frac{1}{m!}\\
&=\sum_{m=2}^{n+1}\sum_{r=1}^{m-1}\frac{\begin{bmatrix}m\\r+1\end{bmatrix}_\lambda}{m!}(r+1)(-1)^{r+1}(1)_{r,\lambda}.
\end{aligned}
\tag{23}
$$

Therefore, by (22) and (23), we obtain the following theorem.

Theorem 2.5. *For $n\geq 1$, we have*

$$
\begin{aligned}
H_{n,\lambda}&=\sum_{r=1}^{n}\frac{(-1)^{r+1}}{r!}(1)_{r,\lambda}H_\lambda(n+1,r)\\
&=\sum_{m=2}^{n+1}\sum_{r=1}^{m-1}\frac{\begin{bmatrix}m\\r+1\end{bmatrix}_\lambda}{m!}(r+1)(-1)^{r+1}(1)_{r,\lambda}.
\end{aligned}
$$

Carlitz defined the degenerate Bernoulli polynomials given by

$$
\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!},\quad (\text{see [4]}).
\tag{24}
$$

When $x=0$, $\beta_{n,\lambda}=\beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers. By (7), (11) and (24), we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left(\sum_{r=0}^{n-2}(-1)^{r+1}\frac{1}{(r+1)!}\left(\beta_{r+1,\lambda}-(1)_{r+1,\lambda}\right)H_{\lambda}(n-1,r)\right)x^{n}\\
&=\sum_{r=0}^{\infty}\frac{(-1)^{r+1}}{(r+1)!}\left(\beta_{r+1,\lambda}-(1)_{r+1,\lambda}\right)\sum_{n=r+2}^{\infty}H_{\lambda}(n-1,r)x^{n}\\
&=x\sum_{r=0}^{\infty}\frac{(-1)^{r+1}}{(r+1)!}\left(\beta_{r+1,\lambda}-(1)_{r+1,\lambda}\right)\sum_{n=r+1}^{\infty}H_{\lambda}(n,r)x^{n}\\
&=x\sum_{r=0}^{\infty}\frac{(-1)^{r+1}}{(r+1)!}\left(\beta_{r+1,\lambda}-(1)_{r+1,\lambda}\right)\frac{1}{1-x}\log_{-\lambda}^{r+1}\left(\frac{1}{1-x}\right)\\
&=\frac{x}{1-x}\sum_{r=0}^{\infty}\frac{1}{(r+1)!}\left(\beta_{r+1,\lambda}-(1)_{r+1,\lambda}\right)\log_{\lambda}^{r+1}(1-x)\\
&=\frac{x}{1-x}\sum_{r=1}^{\infty}\frac{1}{r!}\left(\beta_{r,\lambda}-(1)_{r,\lambda}\right)\log_{\lambda}^{r}(1-x) \qquad (25)\\
&=\frac{x}{1-x}\left(\sum_{r=0}^{\infty}\beta_{r,\lambda}\frac{\log_{\lambda}^{r}(1-x)}{r!}-\sum_{r=0}^{\infty}\frac{(1)_{r,\lambda}}{r!}\log_{\lambda}^{r}(1-x)\right)\\
&=\frac{x}{1-x}\left(\frac{\log_{\lambda}(1-x)}{e_{\lambda}\left(\log_{\lambda}(1-x)\right)-1}-e_{\lambda}\left(\log_{\lambda}(1-x)\right)\right)\\
&=-\frac{\log_{\lambda}(1-x)}{1-x}-x=\sum_{n=1}^{\infty}H_{n,\lambda}x^{n}-x\\
&=\sum_{n=2}^{\infty}H_{n,\lambda}x^{n}.
\end{aligned}$$

By comparing the coefficients on both sides of (25), we get

$$\sum_{r=0}^{n-2}\frac{(-1)^{r+1}}{(r+1)!}\left(\beta_{r+1,\lambda}-(1)_{r+1,\lambda}\right)H_{\lambda}(n-1,r)=H_{n,\lambda},\quad (n\geq 2). \tag{26}$$

On the other hand, by (14), we get

$$\begin{aligned}
&\sum_{r=0}^{n-2}(-1)^{r+1}\left(\beta_{r+1,\lambda}-(1)_{r+1,\lambda}\right)\frac{H_{\lambda}(n-1,r)}{(r+1)!}\\
&=\sum_{r=0}^{n-2}(-1)^{r+1}\left(\beta_{r+1,\lambda}-(1)_{r+1,\lambda}\right)\sum_{m=r+1}^{n-1}\frac{\begin{bmatrix} m\\ r+1\end{bmatrix}_{\lambda}}{m!} \qquad (27)\\
&=\sum_{m=1}^{n-1}\sum_{r=0}^{m-1}\frac{\begin{bmatrix} m\\ r+1\end{bmatrix}_{\lambda}}{m!}(-1)^{r+1}\left(\beta_{r+1,\lambda}-(1)_{r+1,\lambda}\right).
\end{aligned}$$

Therefore, by (26) and (27), we obtain the following theorem.

Theorem 2.6. *For $n \geq 2$, we have*

$$H_{n,\lambda} = \sum_{r=0}^{n-2} \frac{(-1)^{r+1}}{(r+1)!} \left(\beta_{r+1,\lambda} - (1)_{r+1,\lambda}\right) H_\lambda(n-1,r)$$
$$= \sum_{m=1}^{n-1} \sum_{r=0}^{m-1} \frac{\begin{bmatrix} m \\ r+1 \end{bmatrix}_\lambda}{m!} (-1)^{r+1} \left(\beta_{r+1,\lambda} - (1)_{r+1,\lambda}\right).$$

From (10) and noting $\sum_{r=1}^{\infty} rt^r = \frac{t}{(1-t)^2}$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{r=1}^{n} (-1)^r r H^{(r)}_{n-r+1,\lambda} x^n = \sum_{r=1}^{\infty} (-1)^r r \sum_{n=r}^{\infty} H^{(r)}_{n-r+1,\lambda} x^n \\
&= \sum_{r=1}^{\infty} (-1)^r r \sum_{n=1}^{\infty} H^{(r)}_n x^{n+r-1} = \sum_{r=1}^{\infty} (-1)^r r x^{r-1} \frac{\log_{-\lambda}\left(\frac{1}{1-x}\right)}{(1-x)^r} \\
&= \frac{\log_{-\lambda}\left(\frac{1}{1-x}\right)}{x} \sum_{r=1}^{\infty} (-1)^r r \left(\frac{x}{1-x}\right)^r = \frac{\log_{-\lambda}\left(\frac{1}{1-x}\right)}{x} x(x-1) \\
&= \log_{-\lambda}\left(\frac{1}{1-x}\right)(x-1) = x \log_{-\lambda}\left(\frac{1}{1-x}\right) - \log_{-\lambda}\left(\frac{1}{1-x}\right) \\
&= \sum_{n=1}^{\infty} \binom{\lambda-1}{n-1} \frac{(-1)^{n-1}}{n} x^{n+1} - \sum_{n=1}^{\infty} \binom{\lambda-1}{n-1} \frac{(-1)^{n-1}}{n} x^n \qquad (28) \\
&= \sum_{n=2}^{\infty} \binom{\lambda-1}{n-2} \frac{(-1)^n}{n-1} x^n + \sum_{n=2}^{\infty} \binom{\lambda-1}{n-1} \frac{(-1)^n}{n} x^n - x \\
&= \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} x^n \left(n \binom{\lambda-1}{n-2} + (n-1)\binom{\lambda-1}{n-1}\right) - x \\
&= \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} x^n \left(n \binom{\lambda}{n-1} - \binom{\lambda-1}{n-1}\right) - x \\
&= \sum_{n=2}^{\infty} \left(\frac{(-1)^n}{(n-1)} \binom{\lambda}{n-1} - \frac{(-1)^n}{n(n-1)} \binom{\lambda-1}{n-1}\right) x^n - x.
\end{aligned}$$

We also note that

$$\sum_{n=1}^{\infty} \sum_{r=1}^{n} (-1)^r r H^{(r)}_{n-r+1,\lambda} x^n = \sum_{n=2}^{\infty} \sum_{r=1}^{n} (-1)^r r H^{(r)}_{n-r+1,\lambda} x^n - x. \tag{29}$$

Therefore, by (28) and (29), we obtain the following theorem.

Theorem 2.7. *For $n \geq 2$, we have*

$$\sum_{r=1}^{n} (-1)^r r H^{(r)}_{n-r+1,\lambda} = \frac{(-1)^n}{(n-1)} \binom{\lambda}{n-1} - \frac{(-1)^n}{n(n-1)} \binom{\lambda-1}{n-1}.$$

3. Conclusion

This paper has significantly advanced the study of degenerate special numbers and polynomials by introducing and analyzing the generalized degenerate harmonic numbers, $H_\lambda(n,r)$.

We derived several important theorems that provide new expressions and identities for these numbers. Specifically, we showed that $H_\lambda(n,r)$ can be expressed as a finite sum involving both products of binomial coefficients (Theorem 2.1) and the unsigned degenerate Stirling numbers of the first kind (Theorem 2.2). We also established two finite sums that demonstrate fascinating relationships between the generalized degenerate harmonic numbers and the unsigned degenerate Stirling numbers of the first kind, with both summing to a simple integer (n in Theorem 2.3 and 1 in Theorem 2.4). Furthermore, we provided new expressions for the original degenerate harmonic numbers, $H_{n,\lambda}$, in terms of our new generalized numbers, the unsigned degenerate Stirling numbers of the first kind and the degenerate Bernoulli numbers (Theorems 2.5 and 2.6). Finally, we offered a closed-form expression for a sum involving the degenerate hyperharmonic numbers (Theorem 2.7).

Future research could focus on exploring additional identities and combinatorial interpretations for the generalized degenerate harmonic numbers. Another promising avenue is to investigate the relationship between these new numbers and other degenerate polynomials and numbers.

References

[1] Abramowitz, M.; Stegun, I. A. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. For sale by the Superintendent of Documents. National Bureau of Standards Applied Mathematics Series, No. **55**. U. S. Government Printing Office, Washington, DC, 1964.

[2] Benjamin, A. T.; Gaebler, D.; Gaebler, R. *A combinatorial approach to hyperharmonic numbers,* Integers **3** (2003), A15, 9 pp.

[3] Benjamin, A. T.; Preston, G. O.; Quinn, J. J. *A Stirling Encounter with Harmonic Numbers,* Math. Mag. **75** (2002), no. 2, 95-103.

[4] Carlitz, L. *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. **15** (1979), 51-88.

[5] Cheon, G.-S.; El-Mikkawy, M. E. A. *Generalized harmonic numbers with Riordan arrays,* J. Number Theory **128** (2008), no. 2, 413-425.

[6] Comtet, L. *Advanced combinatorics. The art of finite and infinite expansions,* Revised and enlarged edition, D. Reidel Publishing Co., Dordrecht, 1974.

[7] Conway, J. H. ; Guy, R. K. *The book of numbers.* (English summary), Copernicus, New York, 1996.

[8] Dolgy, D. V.; Kim, D. S.; Kim, H.K.; Kim, T. *Degenerate harmonic and hyperharmonic numbers,* Proc. Jangjeon Math. Soc. **26** (2023), no. 3, 259-268.

[9] Jeong, Y.; Kim, D. S. *Identities involving generalized harmonic numbers,* Proc. Jangjeon Math. Soc. **18** (2015), no. 2, 189-199.

[10] Kim, D. S.; Kim T. *Degenerate Sheffer sequences and λ-Sheffer sequences,* J. Math. Anal. Appl. **493** (2021), 124521.

[11] Kim, D. S.; Kim T.; Lee, S.-H. *Combinatorial identities involving harmonic and hyperharmonic numbers,* Adv. Stud. Contemp. Math. (Kyungshang) **23** (2013), no. 3, 393-413.

[12] Kim, H. K.; Lee, D S. *Note on the degenerate r-Dowling polynomials and numbers of the first kind including the generalized degenerate Jindalrae-Stirling numbers and the weak Gaenari polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **33** (2023), no. 4, 305-318.

[13] Kim, T.; Kim, D. S. *Some identities on degenerate harmonic and degenerate higher-order harmonic numbers,* Appl. Math. Comput. **486** (2025), Paper No. 129045, 10 pp.

[14] Kim, T.; Kim, D. S. *Combinatorial identities involving degenerate harmonic and hyperharmonic numbers,* Adv. in Appl. Math. **148** (2023), Paper No. 102535, 15 pp.

[15] Kim, T.; Kim, D. S. *Spivey-type recurrence relations for degenerate Bell and Dowling polynomials,* Russ. J. Math. Phys. **32** (2025), no. 2, 288-296.

[16] Kim, T.; Kim, D. S. *Note on the degenerate gamma function,* Russ. J. Math. Phys. **27** (2020), no. 3, 352–358.

[17] Kim, T.; Kim, D. S.; Kim, H. K. *Some identities of degenerate harmonic and degenerate hyperharmonic numbers arising from umbral calculus,* Open Math. **21** (2023), no. 1, Paper No. 20230124, 12 pp.

[18] Kurt, B. *Notes on the degenerate harmonic numbers and polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **33** (2023), no. 3, 213-219.

[19] Li, C.; Chu, W. *Fifteen series on harmonic numbers and quintic central binomial coefficients,* Discrete Math. Lett. **16** (2025), 15-20.

[20] Luo, L.; Ma, Y.; Kim, T.; Xu, R. *Series involving degenerate harmonic numbers and degenerate Stirling numbers,* Appl. Math. Sci. Eng. **32** (2024), no. 1, Paper No. 2297045, 11 pp.

[21] Roman, S. *The umbral calculus,* Pure and Applied Mathematics **111**, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.

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